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A new velocity–vorticity boundary integral formulation for Navier–Stokes equations

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SUMMARY

In the present work, an indirect boundary integral method for the numerical solution of Navier–Stokes equations formulated in velocity–vorticity dependent variables is proposed. This wholly integral approach, based on Helmholtz's decomposition, deals directly with the vorticity field and gives emphasis to the establishment of appropriate boundary conditions for the vorticity transport equation. The coupling between the vorticity and the vortical velocity fields is expressed by an iterative procedure. The present analysis shows the usefulness of an integral formulation not only in providing a potentially more efficient computational tool, but also in giving a better understanding to the physics of the phenomenon.

KEY WORDS: boundary element method; Green's function; Helmholtz's decomposition; integral equation; Navier–Stokes equations; velocity–vorticity formulation

1. INTRODUCTION

In potential theory, the boundary element method (BEM) has now become a numerical tool for solving many types of boundary value problems. One of the most attractive features of this method is that the dimension of the problem is lowered by one order. Further extension in the application of the BEM has been hampered by the difficulties encountered when dealing with rotational flow, as the vorticity involves domain integrals. In these cases, the boundary-only nature of the approach, which is its most attractive aspect, is lost. Consequently, the BEM may appear disadvantageous when compared with more classical domain schemes, such as the finite difference and finite element methods.

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Among the various attempts to get rid of the entire-domain integrals evaluation, we should mention the dual reciprocity method [1] or the particular integrals technique [2], which transform domain integrals into boundary integrals. However, application of these approaches to the complete Navier–Stokes equations is restricted to small Reynolds numbers [3,4]. Apart from this limited case, the possibility of obtaining accurate and less costly results at all Reynolds numbers is still, up to now, subject to intensive investigations.

In this paper, the authors propose a new general method for solving the Navier–Stokes equations by the BEM using a velocity–vorticity formulation. This approach recasts both the kinematic and kinetic aspects of the problem into integral forms. It is based on the concept of transforming local boundary conditions given on the boundary of a fluid region into global, or integral, conditions taken over it. More specifically, boundary conditions on the velocity assign conditions of an integral type on the vorticity.

The formulation is based on the decomposition of a solenoidal velocity field into two components, one that is irrotational (potential velocity) and the other that is related to the vorticity (vortical velocity). In the proposed method, three principal unknowns can be distinguished: the potential velocity $\mathbf{u}_l$ (the subscript 'l' refers to Laplacian field), the vortical velocity $\mathbf{u}_v$ and the vorticity field ω. Each unknown is computed explicitly by introducing specific distributed singularities on the boundaries of the fluid domain. A convenient integral representation based on an indirect formulation follows from the application of Green's formula. Another integral representation, based on a direct formulation, has not yet been considered but should be preferentially developed and used for solid boundaries that contain corners.

The method presented here is actually an extension of the integro-differential formulation previously proposed by Achard and Canot [5]. The common points lie in the representation of the potential and vortical components of the velocity field. The two approaches differ in the determination of the vorticity field. The present contribution deals with the articulation between $\mathbf{u}_v$ and ω. More specifically, appropriate boundary conditions are converted into integral forms by using distributions of surface singularities; in particular, the vorticity transport equation is found to be subject to integral projection conditions.

Apart from the uncoupled problem governing the potential flow, a system of coupled boundary integral equations for the vortical flow is obtained; this latter must be solved by iterative techniques, even in the linear case. It is, in fact, such a solution procedure, entirely restricted to the boundaries and with volume integrals being involved as forcing terms, which provides the new features of this method. Compared with the usual primitive variables (velocity–pressure) approach this velocity–vorticity formulation introduces additional variables; however, the dimension of the problem is reduced by one. For the sake of clarity, the basic concepts of our approach are introduced for steady state viscous flows. The unsteady state case takes somewhat longer to treat but does not involve additional difficulties.

The numerical simulation of the flow between two concentric cylinders, the inner being in translation, supports the validity and effectiveness of the proposed iterative solver. Computational results are compared with the analytical solution available for low Reynolds number approximation. Hence, the purpose of this study is twofold: (i) to describe a new integral representation together with the associated solution procedure, and (ii) to demonstrate the validity of this formulation by a numerical illustration.

The paper is organized as follows. The next section introduces the mathematical tools by recalling the basic aspects that have led previously to an integro-differential formulation [5]. The benefits concern the representation of the vorticity field and the use of projection conditions; these points are addressed in Section 3. In Section 4, the iterative algorithm, issued from this wholly integral approach, is described. Subsection 4.2 deals with the computational implementation of the method for the case of a two-dimensional steady flow. Finally, the last section is devoted to a few conclusive remarks.

2. REVIEW ON THE INTEGRO-DIFFERENTIAL FORMULATION

Let R be a three-dimensional simply connected region bounded by a surface S and $\mathbf{n}$ its unit outward normal vector. The time-dependent flow of an incompressible fluid is governed by the equation of continuity

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

and the momentum equations, which in the case of isothermal flows and Newtonian fluids are the Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\rho^{-1}\nabla p - \nu \nabla \times \omega + \mathbf{F} \tag{2}$$

Here $\mathbf{u}$ is the flow velocity, t denotes time, p is the pressure, ρ and ν are respectively the density and kinematic viscosity of the fluid, and $\mathbf{F}$ describes the body forces that apply to the whole of a fluid element; the vorticity vector ω is defined by

$$\omega \equiv \nabla \times \mathbf{u} \tag{3}$$

In addition to the above field equations, initial and boundary conditions on the velocity field $\mathbf{u}$ are required. They are respectively

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \tag{4}$$

$$\mathbf{u}|_S = \mathbf{u}_\mathrm{b} \quad \text{with} \quad \int_S \mathbf{u}_\mathrm{b} \cdot \mathbf{n} \, \mathrm{d}S = 0 \tag{5}$$

in which $\mathbf{u}_0$ defines the initial velocity field and $\mathbf{u}_\mathrm{b}$ is a prescribed velocity on the boundary.

Two of the main difficulties inherent in determining the flow of an incompressible fluid, by solving the set of equations (1)–(5) are, first, that the momentum equations (2) have to be solved subject to the continuity constraint (1), and second, that there is no evolution equation for the pressure. A method for overcoming these difficulties is to utilize the concept of vorticity and partition the flow problem into its kinetic and kinematic aspects.

The aspect of potential theory, which is of importance here, is the expression of a vector field in terms of scalar and vector potential fields. Helmholtz's decomposition states that the solenoidal velocity vector can be expressed as

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_v = \nabla\varphi + \nabla \times \psi \tag{6}$$

where φ is the harmonic scalar potential and ψ is a solenoidal vector potential satisfying the Poisson's equation.

Accordingly, the potential velocity component $\mathbf{u}_l$ is governed by the following set of equations:

$$\mathbf{u}_l = \nabla\varphi \quad \text{and} \quad \nabla^2\varphi = 0 \quad \text{in } R \tag{7}$$

$$\mathbf{n}\cdot\nabla\varphi = \mathbf{n}\cdot\mathbf{u}_b \quad \text{on } S \tag{8}$$

Condition (8) expresses the entire attribution to the potential flow of the normal velocity component at the boundary.

In other respects, for a given distribution of ω, the vortical velocity component $\mathbf{u}_v$ is evaluated by a vector Poisson's equation

$$\mathbf{u}_v = \nabla \times \psi \quad \text{and} \quad \nabla^2\psi = -\omega \quad \text{in } R \tag{9}$$

associated with the following boundary conditions:

$$\nabla\cdot\psi\big|_S = 0 \quad \text{and} \quad \mathbf{n}\cdot(\nabla \times \psi)\big|_S = 0 \tag{10}$$

$$\mathbf{n} \times (\nabla \times \psi)\big|_S = \mathbf{n} \times (\mathbf{u}_b - \nabla\varphi\big|_S) \tag{11}$$

Furthermore, taking the curl of each term in the Navier–Stokes equations (2) eliminates the pressure as a dependent variable and gives the vorticity transport equation

$$\frac{\partial\omega}{\partial t} + \nabla \times (\omega \times \mathbf{u}) = \nu\nabla^2\omega \tag{12}$$

in which use of the continuity equation (1) and definition of the vorticity vector (3) have been made.

Using the concept of fundamental solution for elliptic problems, the set of the above differential equations governing the velocity and vorticity fields will be recast into an integral representation on the boundaries of the fluid domain. For this purpose, we apply Green's theorem for scalars and vectors and make use of the three-dimensional free-space fundamental solution

$$G(\boldsymbol{x}, \boldsymbol{x}') \equiv \frac{1}{4\pi s}, \quad \text{with } s = \|\boldsymbol{x} - \boldsymbol{x}'\| \tag{13}$$

$\boldsymbol{x}$ is the source point and $\boldsymbol{x}'$ is the integration (or field) point. Moreover, in an indirect formulation we consider the complementary domain R_c such that the three-dimensional space is $R^3 = R \cup S \cup R_c$, and we define harmonic complementary functions φ_c and ψ_c as fictitious scalar and vector potentials of the velocity field respectively.

The velocity integral representations issued from the previous integro-differential formulation are briefly reviewed below. Details of the mathematical analysis have already been thoroughly described by Achard and Canot [5] to which the reader is referred.

2.1. Potential velocity

For Neumann-type boundary conditions, the scalar potential is generated by a source distribution

$$\sigma = -\mathbf{n}\cdot\nabla(\varphi - \varphi_c)|_S \tag{14}$$

The densities of these sources are governed by a second kind Fredholm equation, which expresses the no-mass transfer condition across the boundaries (Equation (8))

$$\frac{1}{2}\sigma(\boldsymbol{x}) + \int_S \sigma(\boldsymbol{x}')k_n(\boldsymbol{x}, \boldsymbol{x}')\,\mathrm{d}S' = -\mathbf{n}\cdot\mathbf{u}_b(\boldsymbol{x}) \tag{15}$$

$\mathbf{n}(\boldsymbol{x})$ defines the unit outward normal vector at source point $\boldsymbol{x}$ of the boundary S.

The irrotationnal component $\mathbf{u}_I$ of the velocity field can then be deduced, $\boldsymbol{x}$ being taken in the fluid region R

$$\mathbf{u}_I(\boldsymbol{x}) = -\int_S \sigma(\boldsymbol{x}')\mathbf{k}(\boldsymbol{x}, \boldsymbol{x}')\,\mathrm{d}S' \tag{16}$$

with the three-dimensional fundamental solution given by

$$\begin{cases} \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}') \equiv \dfrac{1}{4\pi}\nabla\left(\dfrac{1}{s}\right) = -\dfrac{1}{4\pi}\dfrac{(\boldsymbol{x} - \boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|^3} \\ k_n(\boldsymbol{x}, \boldsymbol{x}') \equiv \mathbf{n}(\boldsymbol{x})\cdot\mathbf{k}(\boldsymbol{x}, \boldsymbol{x}') = -\dfrac{1}{4\pi}\dfrac{\mathbf{n}(\boldsymbol{x})\cdot(\boldsymbol{x} - \boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|^3}, \quad \forall \boldsymbol{x} \in S \end{cases} \tag{17}$$

2.2. Vortical velocity

Similarly, the vortical velocity component $\mathbf{u}_v$ is computed explicitly by introducing specific distributed singularities on the boundaries. We define by

$$\gamma \equiv -\mathbf{n} \times [\nabla \times (\psi - \psi_c)]|_S \tag{18}$$

the densities of a local sheet vortex, which have the dimensions of velocity (refer to Batchelor [6]). The no-slip boundary condition (11), for the total velocity $\mathbf{u}_l + \mathbf{u}_v$, can then be expressed by the following Fredholm vectorial equation of the second kind:

$$\begin{aligned}&\frac{1}{2}\gamma(\boldsymbol{x}) + \mathbf{n}(\boldsymbol{x}) \times \int_S [\gamma(\boldsymbol{x}') \times \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}')]\, \mathrm{d}S' \\ &= -[\mathbf{n} \times (\mathbf{u}_b - \nabla\varphi|_S)](\boldsymbol{x}) - \mathbf{n}(\boldsymbol{x}) \times \int_R [\omega(\boldsymbol{x}') \times \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}')]\, \mathrm{d}\boldsymbol{x}'\end{aligned} \tag{19}$$

The solution of this equation along the boundary S leads to the densities γ of the vectorial source distribution. This requires the knowledge of the vorticity ω throughout the flow field and also the tangential velocity along the boundary. This latter constitutes an external forcing term in the generation of the vorticity at solid boundaries, whereas the former operates as an internal coupling term.

We are actually interested in the curl of the vector potential, which gives $\mathbf{u}_v$ (the vortical component of velocity field)

$$\mathbf{u}_v(\boldsymbol{x}) = \int_R [\omega(\boldsymbol{x}') \times \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}')]\, \mathrm{d}\boldsymbol{x}' + \int_S [\gamma(\boldsymbol{x}') \times \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}')]\, \mathrm{d}S' \tag{20}$$

This relation can be interpreted as a generalization of the well-known Biot–Savart law for distributed vorticity in the region R; the boundary integral represents the contribution from velocity boundary conditions. As can be seen in Equation (20), only the vorticity distribution in the viscous region of the flow contributes to the calculation of the velocity anywhere in the flow, as the integrand in the domain integral vanishes for $\omega = 0$. An obvious advantage of the present formulation follows directly: as often as the viscous regions occupy a small fraction of the entire flow field (as is the case in inertia-dominant flows), a considerable reduction in the size of the computational domain is obtained.

3. VORTICITY FIELD AND THE ASSOCIATED BOUNDARY CONDITIONS

The kinetic aspects of the flow are represented by the vorticity transport equation, which, for a steady incompressible flow, takes the form

$$\nabla^2\omega = \frac{1}{\nu}[\nabla \times (\omega \times \mathbf{u}_l) + \nabla \times (\omega \times \mathbf{u}_v)] \tag{21}$$

to which the following boundary conditions are associated:

$$\nabla \cdot \omega|_S = 0 \tag{22}$$

$$\mathbf{n}\cdot\omega\big|_S = \mathbf{n}\cdot\nabla_\tau \times \mathbf{u}_\mathrm{b} \tag{23}$$

$$\mathbf{n}(\boldsymbol{x})\cdot\int_R [\omega(\boldsymbol{x}') \times \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}')]\,\mathrm{d}\boldsymbol{x}' = -\mathbf{n}(\boldsymbol{x})\cdot\int_S [\gamma(\boldsymbol{x}') \times \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}')]\,\mathrm{d}S' \tag{24}$$

Note that the two first mathematical conditions for the vorticity (Equations (22) and (23)) have already been established by Quartapelle and Valz-Gris [7], whereas the latter is more original (see Achard and Canot [5]) and has the peculiarity of being of integral (non-local) form instead of the usual local boundary conditions. The operator ∇_τ in Equation (23) is the surface gradient operator.

Let us now consider the transformation of this set of equations (21)–(24) into an elliptic problem, which leads to an integral representation of the vorticity field ω. Similarly, as for the scalar φ and vector ψ potential fields, Equation (21) can be written as a Poisson's problem

$$\nabla^2\omega = -f \quad \text{with } f = -\frac{1}{\nu}[\nabla \times (\omega \times \mathbf{u}_\mathrm{l}) + \nabla \times (\omega \times \mathbf{u}_\mathrm{v})] \tag{25}$$

By application of Green's formula, a total vorticity ω_t reads

$$\omega_\mathrm{t}(\boldsymbol{x}) = \begin{Bmatrix} \omega & \text{if } \boldsymbol{x}\in R \\ \frac{1}{2}(\omega + \omega_\mathrm{c}) & \text{if } \boldsymbol{x}\in S \\ \omega_\mathrm{c} & \text{if } \boldsymbol{x}\in R_\mathrm{c} \end{Bmatrix}$$
$$= \left| \begin{array}{l} -\dfrac{1}{4\pi}\displaystyle\int_S [\mathbf{n}(\boldsymbol{x})\cdot(\omega - \omega_\mathrm{c})](\boldsymbol{x}')\cdot\nabla'(1/s)\,\mathrm{d}S' \\ -\dfrac{1}{4\pi}\displaystyle\int_S [\mathbf{n}(\boldsymbol{x}) \times (\omega - \omega_\mathrm{c})](\boldsymbol{x}') \times \nabla'(1/s)\,\mathrm{d}S' \\ -\dfrac{1}{4\pi}\displaystyle\int_S \frac{\mathbf{n}(\boldsymbol{x}) \times [\nabla' \times (\omega - \omega_\mathrm{c})](\boldsymbol{x}')}{s}\,\mathrm{d}S' \\ +\dfrac{1}{4\pi}\displaystyle\int_S \frac{\mathbf{n}(\boldsymbol{x})\cdot[\nabla'\cdot(\omega - \omega_\mathrm{c})](\boldsymbol{x}')}{s}\,\mathrm{d}S' + \dfrac{1}{4\pi}\displaystyle\int_R \frac{\mathbf{f}(\boldsymbol{x}')}{s}\,\mathrm{d}\boldsymbol{x}' \end{array} \right. \tag{26}$$

In the above integral representation, a degree of arbitrariness is involved. Among the various possibilities, we eliminate two of the four involved singularity distributions, such that

$$\mathbf{n} \times (\omega - \omega_\mathrm{c})\big|_S = 0 \quad \text{and} \quad \nabla\cdot(\omega - \omega_\mathrm{c})\big|_S = 0 \tag{27}$$

Thus, a convenient form for the total vorticity follows directly:

$$\omega_{\mathrm{t}}(\boldsymbol{x}) = \left\{ \begin{array}{ll} \omega & \text{if } \boldsymbol{x} \in R \\ \frac{1}{2}(\omega + \omega_{\mathrm{c}}) & \text{if } \boldsymbol{x} \in S \\ \omega_{\mathrm{c}} & \text{if } \boldsymbol{x} \in R_{\mathrm{c}} \end{array} \right\}$$

$$= \frac{1}{4\pi} \int_R \frac{\mathbf{f}(\boldsymbol{x}')}{s} \, \mathrm{d}\boldsymbol{x}' + \frac{1}{4\pi} \int_S \frac{\beta(\boldsymbol{x}')}{s} \, \mathrm{d}S' - \int_S \delta(\boldsymbol{x}')\mathbf{k}(\boldsymbol{x}, \boldsymbol{x}') \, \mathrm{d}S' \tag{28}$$

In Equation (28), the first integral represents the contribution of the convective process to the vorticity generation. The two boundary integrals give the contribution of the vorticity boundary condition. Note that two specific surface singularities associated with ω have been introduced, namely, a normal doublet density δ

$$\delta \equiv -\mathbf{n} \cdot (\omega - \omega_{\mathrm{c}})|_S \tag{29}$$

and a vectorial source β

$$\beta \equiv -\mathbf{n} \times [\nabla \times (\omega - \omega_{\mathrm{c}})]|_S \tag{30}$$

Let us now examine the different boundary conditions associated with the vorticity. Taking the normal projection of Equation (28) and using condition (23), we obtain the following boundary integral equation:

$$\frac{1}{2}\delta(\boldsymbol{x}) + \int_S [\delta(\boldsymbol{x}')k_n(\boldsymbol{x}, \boldsymbol{x}')] \, \mathrm{d}S' = \frac{1}{4\pi} \int_R \frac{\mathbf{n} \cdot \mathbf{f}(\boldsymbol{x}')}{s} \, \mathrm{d}\boldsymbol{x}' + \frac{1}{4\pi} \int_S \frac{\mathbf{n} \cdot \beta(\boldsymbol{x}')}{s} \, \mathrm{d}S' - \mathbf{n} \cdot \nabla_\tau \times \mathbf{u}_{\mathrm{b}} \tag{31}$$

Hence, it appears that the boundary condition for the normal component of ω (Equation (23)) leads to a well-posed second kind Fredholm equation in which $\mathbf{n} \cdot \nabla_\tau \times \mathbf{u}_{\mathrm{b}}$ operates as an external forcing term. The solenoidal property satisfied by ω on boundary S (Equation (22)) gives an integral projection equation [5]

$$\int_R \mathbf{f}(\boldsymbol{x}') \cdot \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}' = -\int_S \beta(\boldsymbol{x}') \cdot \mathbf{k}(\boldsymbol{x}, \boldsymbol{x}') \, \mathrm{d}S' \tag{32}$$

Finally, as the normal component of $\mathbf{u}_{\mathrm{v}}$ has been set to zero at the boundary (Equation (10)), we obtain a second integral projection equation, which results immediately from Equation (24)

$$\int_R [\mathbf{f}(\boldsymbol{x}') \cdot \mathbf{l}(\boldsymbol{x}, \boldsymbol{x}')] \, \mathrm{d}\boldsymbol{x}' + \int_S [\beta(\boldsymbol{x}') \cdot \mathbf{l}(\boldsymbol{x}, \boldsymbol{x}')] \, \mathrm{d}S' - \int_S [\delta(\boldsymbol{x}') \cdot m(\boldsymbol{x}, \boldsymbol{x}')] \, \mathrm{d}S'$$

$$+ \mathbf{n}(\boldsymbol{x}) \cdot \int_S [\gamma(\boldsymbol{x}') \times \mathbf{k}_{\mathrm{t}}(\boldsymbol{x}, \boldsymbol{x}')] \, \mathrm{d}S' = 0 \tag{33}$$

In the above equation, the kernels $\mathbf{l}(\boldsymbol{x}, \boldsymbol{x}')$ and $m(\boldsymbol{x}, \boldsymbol{x}')$ are defined by

$$\begin{cases} \mathbf{l}(\boldsymbol{x}, \boldsymbol{x}') = \dfrac{1}{4\pi} \displaystyle\int_R \dfrac{1}{\|\boldsymbol{x}'' - \boldsymbol{x}'\|} \cdot [\mathbf{k}_t(\boldsymbol{x}, \boldsymbol{x}'') \times \mathbf{n}(\boldsymbol{x})] \, \mathrm{d}\boldsymbol{x}'' \\ m(\boldsymbol{x}, \boldsymbol{x}') = \displaystyle\int_R \mathbf{k}(\boldsymbol{x}'', \boldsymbol{x}') \cdot [\mathbf{k}_t(\boldsymbol{x}, \boldsymbol{x}'') \times \mathbf{n}(\boldsymbol{x})] \, \mathrm{d}\boldsymbol{x}'' \end{cases} \tag{34}$$

with

$$\begin{cases} \mathbf{k}(\boldsymbol{x}'', \boldsymbol{x}') = -\dfrac{1}{4\pi} \dfrac{(\boldsymbol{x}'' - \boldsymbol{x}')}{\|\boldsymbol{x}'' - \boldsymbol{x}'\|^3} \\ \mathbf{k}_t(\boldsymbol{x}, \boldsymbol{x}'') \equiv \mathbf{n}(\boldsymbol{x}) \times [\mathbf{k}(\boldsymbol{x}, \boldsymbol{x}'') \times \mathbf{n}(\boldsymbol{x})] = \dfrac{1}{4\pi} \dfrac{\mathbf{n}(\boldsymbol{x}) \times [\mathbf{n}(\boldsymbol{x}) \times (\boldsymbol{x} - \boldsymbol{x}'')]}{\|\boldsymbol{x} - \boldsymbol{x}''\|^3} \end{cases} \tag{35}$$

We remark that Equation (33) includes the densities γ of a vortex sheet associated with the unknown $\mathbf{u}_v$. Accordingly, this last condition provides a linkage between the regular ω and singular γ vortical fields. Besides, note that the involved kernels $\mathbf{l}$ and m are shape factors that depend only on the geometry of the fluid region R; for a fixed fluid domain they are consequently determined only once at the very outset of the computational process.

The system obtained for the three principal unknowns $\mathbf{u}_l$, $\mathbf{u}_v$ and ω involves integral equations in which the only unknowns are located on the boundaries of the fluid domain. Henceforth, from the theory presented along this section and the previous one, it is possible to give explicitly a coherent procedure for solving three-dimensional incompressible flow problems.

4. COMPUTATIONAL TECHNIQUE

4.1. Iterative procedure

Equations (15), (16), (19), (20), (28) and (31)–(33) form a complete set defining $\mathbf{u}_l$, $\mathbf{u}_v$ and ω at all points in the flow domain R and $(\sigma, \gamma, \delta, \beta)$ on its boundary S. In general, the implementation of the solution will be of numerical nature. Figure 1 summarizes the solution procedure and illustrates the coupling between the vorticity and the velocity fields. It is worth pointing out that apart from the uncoupled problem governing the potential flow, a system of coupled boundary integral equations for the vortical flow is obtained, which must be solved by iterative techniques, even in the linear case. The new feature of our method lies in the fact that this solution procedure is restricted only over the boundaries: the involved volume integrals act, indeed, as forcing terms.

Consequently, the contribution of the inviscid part of the flow to the computation of ω anywhere in the flow is zero, as it was in the calculation of $\mathbf{u}_v$. Only the values of $\mathbf{u}$ on S that are prescribed by the boundary conditions in the viscous region of the flow are needed for the calculation of ω. The solution field can then be confined to the viscous region of the flow. Since these latter ones are generally embedded in a much larger inviscid region, a considerable reduction in the size of the flow domain may then be achieved in the actual computations.

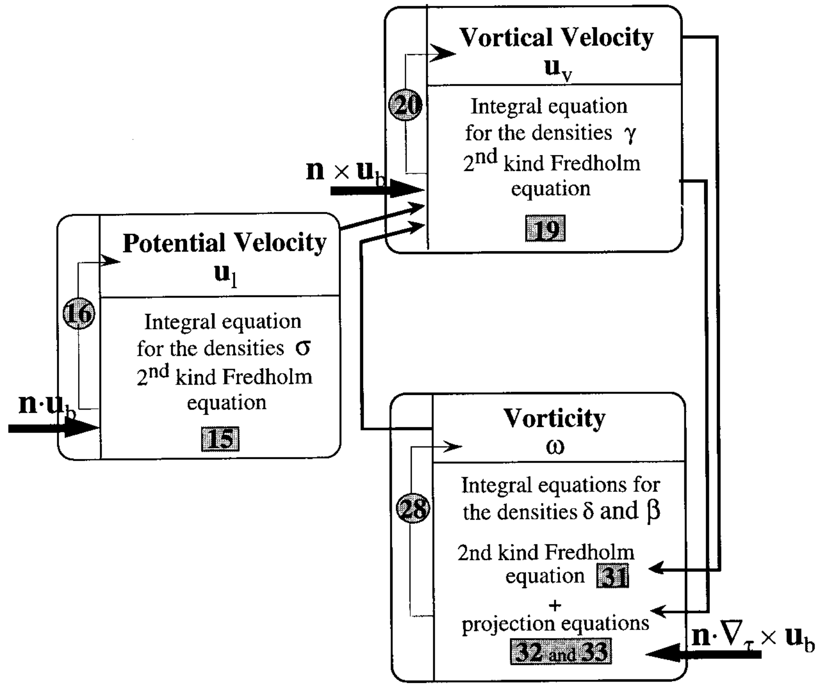

Figure 1. Scheme of the computational process.

In this section, the numerical implementation of the proposed method is presented. The solution of the issued equation set follows basically the steps described in the algorithm given in Figure 2. Note, in this figure, that only $\mathbf{u}_v$ and ω are altered at each iteration; the irrotational field $\mathbf{u}_l$ or other terms depending on geometry are computed only once, stored and used in each iteration. Details of the calculations have already been thoroughly described by Machane [8], to which the reader is referred. Simplifications of the involved equations for a two-dimensional flow configuration are also addressed in this same reference.

We check the convergence of the iterative process by computing the discrepancy E_c between two successive estimations k and $(k+1)$ of the vorticity values

$$E_c = \frac{\sum_{i=1}^{M} \sum_{j=1}^{N} [\omega_{k+1}(i,j) - \omega_k(i,j)]^2}{\sum_{i=1}^{M} \sum_{j=1}^{N} [\omega_{k+1}(i,j)]^2} \tag{36}$$

where $M \times N$ defines the meshing of the computational domain. Thus, the iterative process is terminated when two consecutive approximate solutions of the vorticity field ω_k and ω_{k+1} are found to satisfy some convergence criterion within a specified accuracy parameter (tolerance) ε.

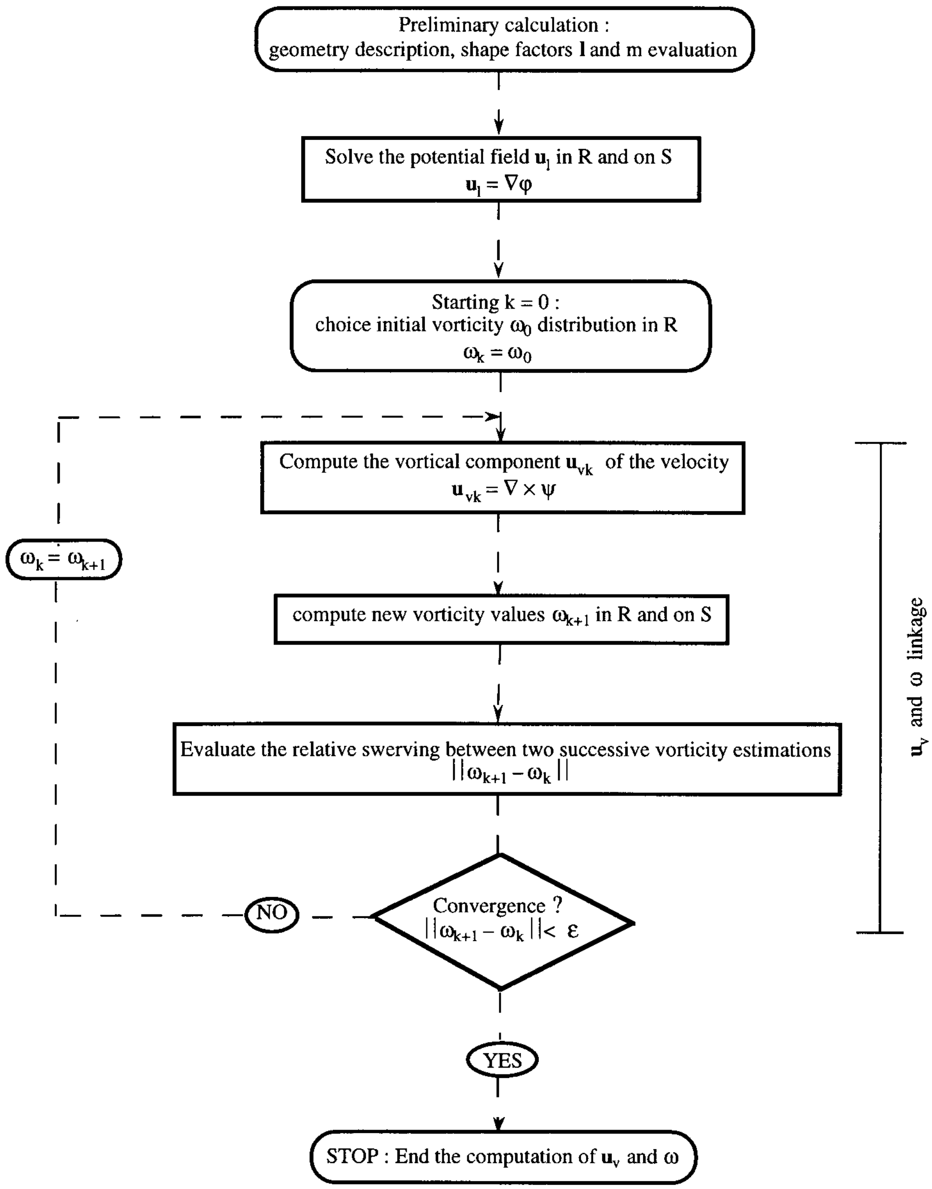

Figure 2. Iterative algorithm computing the velocity and vorticity fields.

4.2. Numerical implementation on a test problem

A first test problem was chosen in order to demonstrate the validity and effectiveness of our approach. This application constitutes only a first validation step of the proposed method; no extensive investigation of the above properties and no detailed comparison with other methods are attempted here.

In the present application, the geometries are kept relatively simple and only flows generated by uniform velocity boundary conditions within circular boundaries are considered. The problem concerns the viscous flow in a two-dimensional space around a circular cylinder, sliding at a constant speed, bounded by a fixed rigid cylinder (see Figure 3).

In the case of a low-Reynolds numbers approximation, an analytical solution is available for this problem (refer to Berker [9]). It is given by the following expressions for the streamfunction:

$$\psi(r, \theta) = \frac{1}{2}\mathbf{U}r \sin\theta \frac{2(a_1^2 + a_2^2)\log\left(\frac{r}{a_2}\right) + \left(\frac{a_2^2}{r^2} - 1\right)(r^2 + a_1^2)}{(a_1^2 + a_2^2)\log\left(\frac{a_1}{a_2}\right) - (a_1^2 - a_2^2)} \tag{37}$$

and the associated potential field

$$\varphi(r, \theta) = -\mathbf{U}\cos\theta \frac{a_1^2}{(a_1^2 - a_2^2)}\left(\frac{a_2^2}{r} + r\right) \tag{38}$$

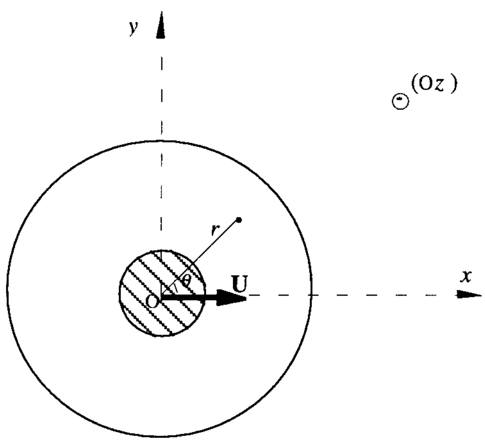

Figure 3. Test example: slow viscous flow between two concentric cylinders. Inner cylinder (moving): radius $a_1 = R_{\min}$; outer cylinder (fixed): radius $a_2 = R_{\max}$.

where (r, θ) defines a polar co-ordinates system, a_1 and a_2 are the radius of the inner and the outer cylinder respectively.

The computations were carried out following the algorithm described in Figure 2. Boundary conditions and the flow region are shown in Figure 3. A rectangular computational domain in the (r, θ) plane was used for the calculations. The computational non-staggered mesh consists of M and N elements in the r- and θ-directions respectively. Special attention was paid to mesh refinement around the moving boundary, where the development of vortices was expected. Cubic continuous geometrical boundary elements and linear triangular internal cells were used. The field functions are described by linear polynomial approximation (C^0 shape basis functions).

Numerical calculations for low Re number values (<20) were performed, where $Re = 2a_1 U/\nu$, with U the velocity of the inner cylinder and a_1 its radius. In each case, the convergence

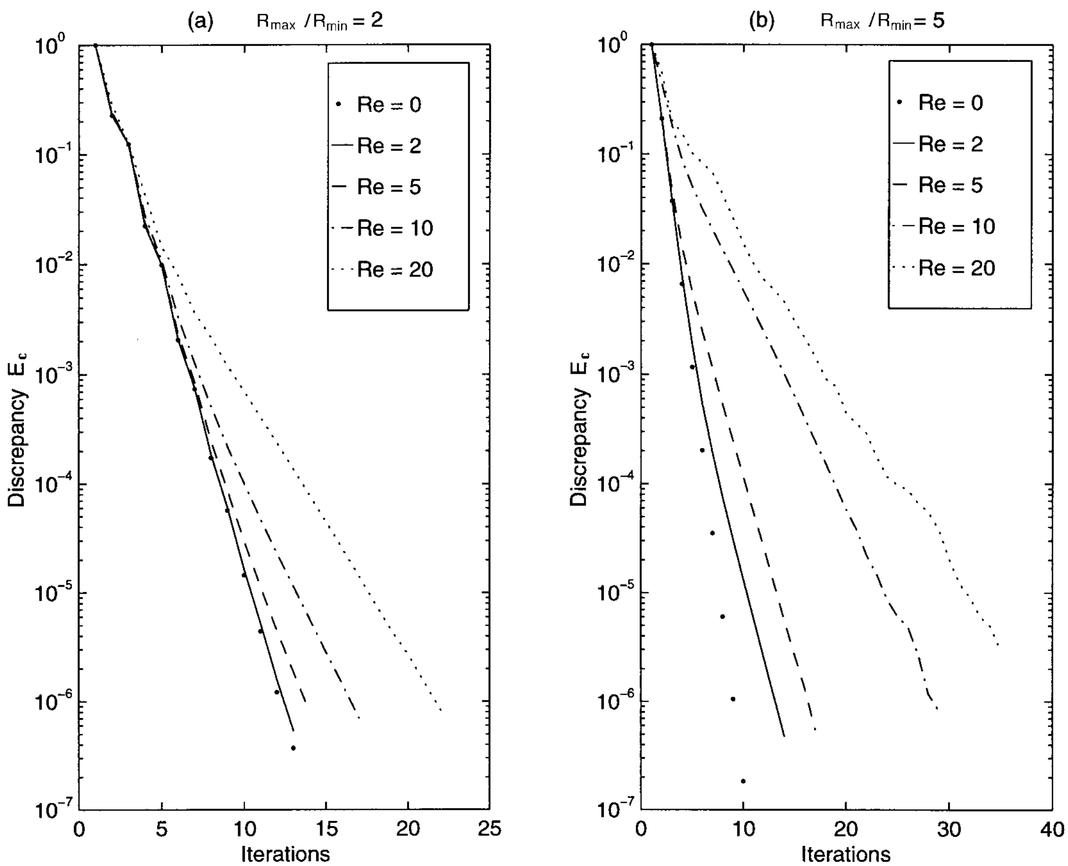

Figure 4. Convergence of the iterative process: (a) $M \times N = 10 \times 40$, (b) $M \times N = 20 \times 40$.

criterion was set to $\varepsilon = 10^{-6}$ and we used the same number of discretization points. All the runs were carried out on a HP/9000 Mod. 715 workstation equipped with 32 MB of RAM with peak performance equal to 50 Mips. The CPU time per iteration is about 400 s for a meshing $M \times N = 20 \times 40$. The best convergence speed is obtained with a relaxation parameter between 0.2 and 0.4. Figure 4 gives some information about the behaviour of the iterative process; the convergence history for different *Re* values is shown. For very small *Re* numbers, the convergence is quite rapid. In Figures 5–7, we present our results, as well as the analytical solution. Computed equivelocity and equivorticity contours are compared in Figures 5 and 6 in which R_{min} and R_{max} define the radius of the inner and the outer cylinder respectively. Vorticity distribution on the boundary of the moving cylinder is given in Figure 7. Note that when the Reynolds numbers decrease towards zero, the calculations converge smoothly to the analytical approximation.

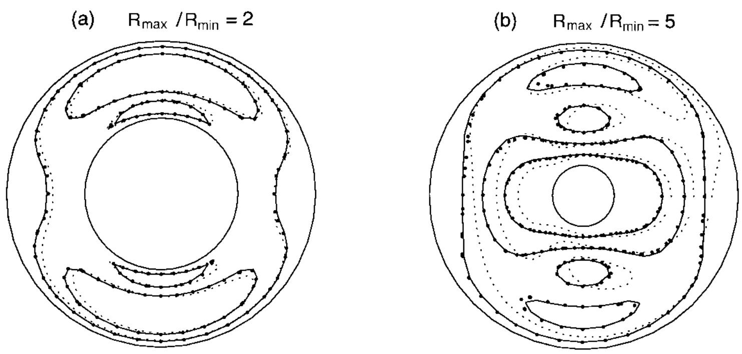

Figure 5. Velocity field **u**—equivelocity contours. — Analytical solution; • • •, numerical solution $Re = 0$; · · · numerical solution $Re = 10$.

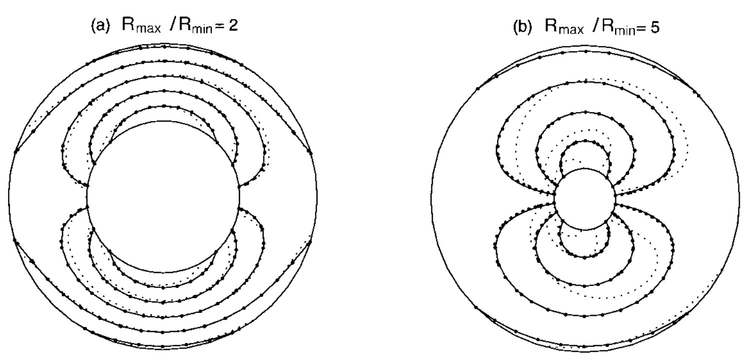

Figure 6. Vorticity field ω—equivorticity contours. — Analytical solution; • • •, numerical solution $Re = 0$; · · · numerical solution $Re = 10$.

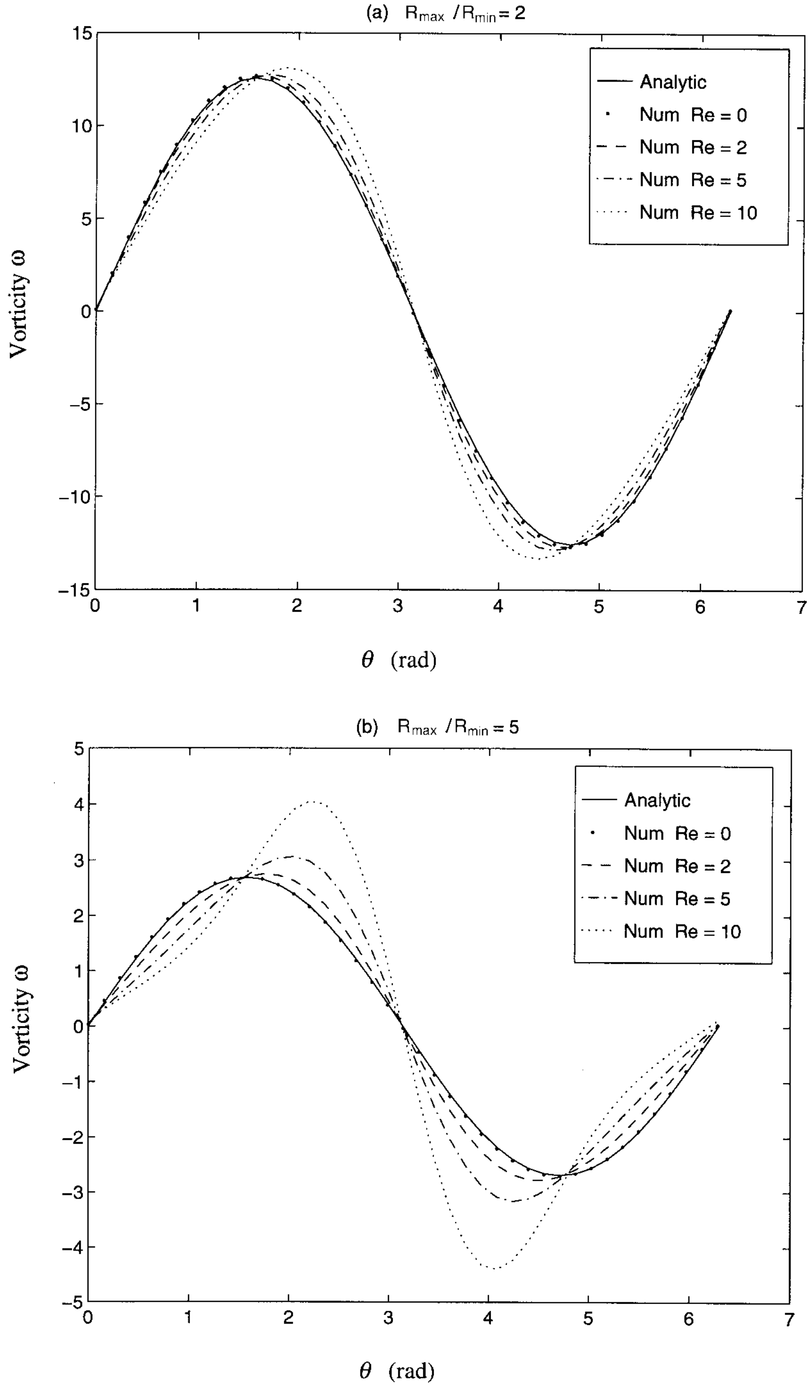

Figure 7. Surface vorticity distribution on the inner cylinder.

The present application, although dealing with a relatively simplified problem, was chosen in order to demonstrate the validity and the effectiveness of our approach. The numerical results reported above give some valuation of the convergence speed as well as the efficiency of this new method, which employs integral projection conditions. Regarding the short iterative process needed to achieve a quite good convergence (see Figure 4), our new method proves to be successful as well as computationally attractive.

5. CONCLUSION

In this paper, a new BEM has been proposed and established to compute incompressible viscous fluid flow problems. A convenient wholly integral form of the Navier–Stokes equations is obtained by using the velocity and vorticity as dependent variables. One original feature of the proposed method lies in the conversion of appropriate boundary conditions, for the considered variables, into boundary integral equations involving the distribution of surface singularities. It is worthwhile mentioning that, in the present formulation, the solution procedure is an iterative technique, where the unknowns are located only on the boundaries and the volume integrals are involved as forcing terms. In other words, the vortical field is constructed progressively through an iterative process involving distribution of specific singularities located only on the boundaries of the fluid domain. Although our approach includes more dependent variables compared with other primitive variables formulations, most of the advantages of BEM techniques are maintained: the resolution technique is entirely restricted to the boundaries and the problem dimension is thus reduced by one order.

Extensions to transient flow problems can be derived for the complete vorticity transport equation from the second Green's identity using the time-dependent fundamental solution [10]. Subsequently, an integral representation equivalent to the steady case still can be obtained. This will require an initial vorticity distribution. Though analogous specific surface singularities can be introduced, nevertheless the equations governing their densities are obviously a little bit more complex. Meanwhile, the derived solution procedure will not introduce much more additional numerical complexities.

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